M321/H

S 1975 Spec

THE OPEN UNIVERSITY Specimen

Third Level Course Examination 1974

R - 1974

PARTIAL DIFFERENTIAL EQUATIONS OF real APPLIED MATHEMATICS

Thursday, 31st October, 1974 **2.30 p.m. – 5.30 p.m.**

Time allowed: 3 hours

You should attempt **NOT MORE THAN THREE** questions from Section A, and **NOT MORE THAN TWO** questions from Section B. Section A carries about 40% of the marks.

You may answer questions in any order, writing your answers in the answer book(s) provided. At the end of the examination, remember to write your name, student number and examination number on the answer book(s) – **failure to do so will mean that your paper(s) cannot be identified.**

Section A

Answer no more than THREE questions from this section.

All questions in this section have equal marks.

Question 1

Write down the function which satisfies Laplace's equation in a square and which takes the value 1 on all four sides.

Show, by using arguments of symmetry, that if $u(x, y)$ satisfies Laplace's equation in a square, whose centre is at the origin, and takes the value 0 on three sides of the square and 1 on the fourth, then $u(0, 0) = \frac{1}{4}$. Explain why your approach is valid.

Hence, by finding the solution $u(x, y)$ throughout the square, verify that

$$\frac{\pi}{16} = \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \frac{1}{e^{(2r+1)\pi/2} + e^{-(2r+1)\pi/2}}$$

Question 2

Show that for a scalar function u with domain R^3

$$\operatorname{div}(u \operatorname{grad} u) = (\operatorname{grad} u)^2 + u\nabla^2 u,$$

and using the Divergence Theorem, or otherwise, deduce that if

(a) $\nabla^2 u = -1$ in a volume D,

(b) at the bounding surface, S, of D either $u = 0$ on all S or $\partial u/\partial n = 0$ on all S,

then

$$\iiint_D u \, dV > 0 \, .$$

Question 3

Find the solution of the problem

$$L[w] \equiv \frac{d^2 w}{dx^2} - w = -x \qquad 0 < x < 1 \, ,$$

$$w(0) = w(1) = 0 \, ,$$

in terms of the eigenfunctions of L or otherwise.

Section A — *continued*

Question 4

The following problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 1, 0 < y < \tfrac{1}{3},$$

$$u(x, 0) = x \qquad 0 \leqslant x \leqslant 1,$$

$$u(0, y) = 0 \qquad 0 \leqslant y \leqslant \tfrac{1}{3},$$

$$u(1, y) = 1 \qquad 0 \leqslant y \leqslant \tfrac{1}{3},$$

$$\frac{\partial u}{\partial y}(x, \tfrac{1}{3}) = 0 \qquad 0 \leqslant x \leqslant 1,$$

is to be solved by the finite-difference method using a uniform square mesh of side $\frac{1}{3}$.

What are the numerical schemes when the derivative $\partial u/\partial y$ is replaced by

(a) a forward-difference formula?

(b) a central-difference formula?

If the numerical scheme in case (b) is put in the form

$$A\mathbf{u} = \mathbf{b},$$

where A is a matrix whose diagonal elements are 1 and $\mathbf{b}$ is a vector whose elements are known, can the simple iterative scheme given by

$$\mathbf{u}^{(n)} = (I - A)\,\mathbf{u}^{(n-1)} + \mathbf{b}$$

be used to solve the equations?

Section B

Answer no more than TWO questions from this section.

All questions in this section have equal marks.

Question 5

A long elastic string is moving with velocity v in the x-direction; it passes through two small rings fixed to the x-axis at the origin and $x = l$. By considering the equation of motion in a frame of reference moving with the string, show that the transverse motion of the string, for $0 \leqslant x \leqslant l$, is described by the equations

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 y}{\partial x^2} - \frac{2v}{c^2}\frac{\partial^2 y}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0 \, ,$$

$$y(0, t) = y(l, t) = 0 \, ,$$

where c is the speed at which small disturbances propagate along the stationary string.

If $v < c$, show that the natural frequencies of vibration are

$$\omega_n = \frac{n\pi c(1 - v^2/c^2)}{l}$$

and that at each of these frequencies the corresponding solution will be proportional to the real part of

$$y_n(x, t) = \exp i\left(\frac{n\pi}{l}(v/c + 1)\{x + (c - v)t\}\right) - \exp i\left(\frac{n\pi}{l}(v/c - 1)\{x - (c + v)t\}\right) .$$

Interpret this equation in terms of wave motion.

(No marks will be given for simply substituting y_n into the differential equation.)

Question 6

For $0 \leqslant x \leqslant \pi$ show that if α is not an integer then

$$\cos \alpha x = \frac{\sin \alpha\pi}{\pi\alpha} + \frac{\sin \alpha\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha + n} + \frac{1}{\alpha - n}\right)\cos nx \, ,$$

and deduce that

$$\pi \cot \pi\alpha - \frac{1}{\alpha} = \sum_{n=1}^{\infty}\left(\frac{1}{\alpha + n} + \frac{1}{\alpha - n}\right) . \qquad (1)$$

By showing that

$$\lim_{\varepsilon \to 0}\int_{\varepsilon}^{z}\left(\pi \cot \pi\alpha - \frac{1}{\alpha}\right)d\alpha = \ln \frac{\sin \pi z}{\pi z} \, ,$$

deduce that for $|z| < 1$,

$$\frac{\sin \pi z}{\pi z} = (1 - z^2)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right)\cdots\left(1 - \frac{z^2}{n^2}\right)\cdots .$$

(You may assume that the series in equation (1) is uniformly convergent on the interval $[-p, p]$ for any $p \in (0, 1)$.)

Section B — *continued*

Question 7

Write the equation

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + 2u = F(x)$$

in self-adjoint form.

Show that the influence function is given by

$$R(x, \xi) = e^{-(x+\xi)} \sin(x - \xi) ,$$

and obtain the solution satisfying $u(0) = a$, $u'(0) = b$ in terms of the influence function.

Deduce that if F is bounded on $[0, \infty)$ then all solutions satisfying the given initial conditions are bounded on $[0, \infty)$. Suggest an argument to show that if in addition $F(x) \to 0$ as $x \to \infty$ then $u(x) \to 0$ in this limit.

Question 8

Classify the equation

$$(n-1)^2 \frac{\partial^2 u}{\partial t^2} - x^{2n} \frac{\partial^2 u}{\partial x^2} = n x^{2n-1} \frac{\partial u}{\partial x} \qquad (x, t) \in R \times R^+, \quad n \neq 1.$$

Reduce it to standard (canonical) form and write down the general solution. Hence show that the solution which satisfies the initial conditions

$$u(x, 0) = F(x) , \qquad \frac{\partial u}{\partial t}(x, 0) = G(x) \qquad x \in R,$$

is

$$u(x, t) = \tfrac{1}{2}\{F(w_+) + F(w_-)\} + \frac{\beta}{2} \int_{w_-}^{w_+} \frac{G(z)}{z^n}\, dz$$

where $\beta = 1 - n$, $w_\pm = (x^\beta \pm t)^{1/\beta}$.

Section B — *continued*

Question 9

For the diffusion equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

Lees' scheme is defined as

$$(1 + r)\, u_{p,\,q+1} - r u_{p-1,\,q+1} = (1 - r)\, u_{p,\,q} + r u_{p+1,\,q},$$

where $r = k/h^2$ is constant and k and h are the mesh spacings in the t- and x-directions respectively. Using von Neumann's method, find the conditions under which Lees' scheme is stable.

(N.B. If z_1, z_2 and ξ are complex numbers such that $\xi = z_1/z_2$ then $|z_1| = |\xi|\,|z_2|$.)

What conditions must be imposed on h and k in order that the resulting numerical solution is convergent to the solution of the differential equation?